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Poisson-Lie T-plurality of three-dimensional conformally invariant sigma models

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Abstract: Starting from the classification of 6-dimensional Drinfeld doubles and their decomposition into Manin triples we construct 3-dimensional Poisson-Lie T-dual or more precisely T-plural sigma models. Of special interest are those that are conformally invariant. Examples of models that satisfy vanishing β-function equations with zero dilaton are presented and their duals are calculated. It turns out that for "traceless" dual algebras they satisfy the β-function equations as well but usually with rather nontrivial dilaton. We also present explicit examples of several kinds of obstacles and difficulties present in construction of quantum dual models. Such concrete examples might be helpful in further development and improvement of quantum version of Poisson-Lie T-duality.

Keywords: Sigma Models, String Duality.

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Contents

1. Introduction

Few years ago, the problem of T-duality of sigma models for the nonabelian groups was solved on the classical (i.e. non-quantum) level. As explained in the paper [1] (see also [2]), the most apropriate structures for formulation of Poisson-Lie T-dual sigma models are Drinfeld doubles. They are connected Lie groups D such that their Lie algebra $\mathcal{D}$ admits a decomposition into two maximally isotropic subalgebras $\mathcal{G}, \tilde{\mathcal{G}}$. It means that the dimension of the Drinfeld double must be even and the dimension of the sigma models is equal to the half of the dimension of the double.

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The classification of the two-dimensional models was given in the papers [3] and [4]. Examples of three-dimensional dual models can be found e.g. in [5, 6, 7] but a classification of three-dimensional models is rather complicated. That's why we have restricted to the models that are conformally invariant and do not produce gravitational anomaly.

In our previous work [8] we have performed complete classification of real 6-dimensional Drinfeld doubles including all possible decompositions into the Manin triples and obtained 22 non-isomorphic classes. For all of them we can construct a variety of models given by three-dimensional subspaces in $\mathcal{D}$.

In the following sections, we firstly recall the definitions of Manin triple and Drinfeld double and briefly explain the construction of dual sigma models. Then we shall present explicit forms of the sigma models that satisfy the vanishing β-function equations with the zero dilaton field and their duals.

2. Drinfeld doubles and Manin triples

The Drinfeld double D is defined as a connected Lie group such that its Lie algebra $\mathcal{D}$ equipped by a symmetric ad-invariant nondegenerate bilinear form $\langle\,.,.\rangle$ can be decomposed into a pair of subalgebras $\mathcal{G}$, $\tilde{\mathcal{G}}$ maximally isotropic with respect to $\langle\,.,.\rangle$ and $\mathcal{D}$ as a vector space is the direct sum of $\mathcal{G}$ and $\tilde{\mathcal{G}}$. This ordered triple of algebras $(\mathcal{D},\mathcal{G},\tilde{\mathcal{G}})$ is called Manin triple.

One can see that the dimensions of the subalgebras are equal and that bases $\{X_i\}$, $\{\tilde{X}^i\}$, $i = 1, 2, 3$ in the subalgebras can be chosen so that

$$\langle X_i, X_j\rangle = 0\,, \qquad \langle X_i, \tilde{X}^j\rangle = \langle \tilde{X}^j, X_i\rangle = \delta_i^j\,, \qquad \langle \tilde{X}^i, \tilde{X}^j\rangle = 0\,. \tag{2.1}$$

Due to the ad-invariance of $\langle\,.,.\rangle$ the algebraic structure of $\mathcal{D}$ is determined by the structure of the maximally isotropic subalgebras because in the basis $\{X_i\},\{\tilde{X}^i\}$ the Lie bracket is given by

$$\begin{aligned} [X_i, X_j] &= f_{ij}{}^k X_k\,, \qquad [\tilde{X}^i, \tilde{X}^j] = \tilde{f^{ij}}{}_k \tilde{X}^k\,,\\ [X_i, \tilde{X}^j] &= f_{ki}{}^j \tilde{X}^k + \tilde{f^{jk}}{}_i X_k\,. \end{aligned} \tag{2.2}$$

It is clear that to any Manin triple $(\mathcal{D},\mathcal{G},\tilde{\mathcal{G}})$ one can construct the dual one by interchanging $\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$, i.e. interchanging the structure coefficients $f_{ij}{}^k \leftrightarrow \tilde{f^{ij}}{}_k$. All properties of Lie algebras (the nontrivial being the Jacobi identities) remain to be satisfied. On the other hand for given Drinfeld double more than two Manin triples can exist and there are many examples of that. Two Drinfeld doubles are isomorphic if they have isomorphic algebraic structure and there is an isomorphism transforming one ad-invariant bilinear form to the other.

Here is the list of nonisomorphic real six-dimensional doubles (see [8]) and their decomposition into the Manin triples (for the notation see appendix B):

Theorem. *Any 6-dimensional real Drinfeld double belongs just to one of the following 22 classes and allows decomposition into all Manin triples listed in the class and their duals* $(\mathcal{G} \leftrightarrow \tilde{\mathcal{G}})$. *If the class contains parameter a or b, the Drinfeld doubles with different values of this parameter are non-isomorphic.*

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DD1 $(9|5|b) \cong (8|5.ii|b) \cong (7_0|5.ii|b),\ b > 0,$

DD2 $(8|5.i|b) \cong (6_0|5.iii|b),\ b > 0,$

DD3 $(7_a|7_{1/a}|b) \cong (7_{1/a}|7_a|b),\ a \geq 1, b \in \mathbf{R} - \{0\},$

DD4 $(6_a|6_{1/a}.i|b) \cong (6_{1/a}.i|6_a|b),\ a > 1, b \in \mathbf{R} - \{0\},$

DD5 $(9|1),$

DD6 $(8|1) \cong (8|5.iii) \cong (7_0|5.i) \cong (6_0|5.i) \cong (5|2.ii),$

DD7 $(7_0|4|b) \cong (4|2.iii|b) \cong (6_0|4.i| - b),\ b \in \mathbf{R} - \{0\},$

DD8 $(3|3.i|b),\ b \in \mathbf{R} - \{0\},$

DD9 $(7_a|1) \cong (7_a|2.i) \cong (7_a|2.ii),\ a > 1,$

DD10 $(6_a|1) \cong (6_a|2) \cong (6_a|6_{1/a}.ii) \cong (6_a|6_{1/a}.iii),\ a > 1,$

DD11 $(6_0|1) \cong (6_0|5.ii) \cong (5|1) \cong (5|2.i),$

DD12 $(6_0|2) \cong (6_0|4.ii) \cong (4|1) \cong (4|2.i) \cong (4|2.ii),$

DD13 $(3|1) \cong (3|2) \cong (3|3.ii) \cong (3|3.iii),$

DD14 $(7_a|1) \cong (7_a|2.i) \cong (7_a|2.ii),\ 0 < a < 1,$

DD15 $(7_0|1),$

DD16 $(7_0|2.i),$

DD17 $(7_0|2.ii),$

DD18 $(7_1|1) \cong (7_1|2.i) \cong (7_1|2.ii),$

DD19 $(2|1),$

DD20 $(2|2.i),$

DD21 $(2|2.ii),$

DD22 $(1|1).$

One can see that for many Drinfeld doubles there are several decompositions into Manin triples. The (non-isomorphic) Manin triples differing only by the value of the parameter b can be transformed one into another by the rescaling of $\langle .,. \rangle$.

As we will see in the next section to each Manin triple one can write down a sigma model on the group G (corresponding to the 1st subalgebra $\mathcal{G}$ in the Manin triple) such that its equations of motion can be, at least in the vicinity of the group unit, written also in the form

$$\langle \partial_\pm l l^{-1}, \mathcal{E}^\pm \rangle = 0 \tag{2.3}$$

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where $l : \mathbf{R}^2 \to D$ and $\mathcal{E}^+, \mathcal{E}^-$ are three-dimensional subspaces of $\mathcal{D}$ [1]. Since this form doesn't depend on the choice of Manin triple, all these models (for fixed $\mathcal{E}^\pm$) are equivalent. Moreover the scaling of $\langle \,.,. \rangle$ does not change the equations of motion (2.3) and consequently all models corresponding to (non-isomorphic) Drinfeld doubles with different choices of the scaling parameter b are equivalent as well.

3. Construction of dual sigma models

Let $\mathcal{E}^+$ be an n-dimensional subspace of $\mathcal{D}$ and $\mathcal{E}^-$ its orthogonal complement with respect to $\langle .,. \rangle$ such that $\mathcal{E}^+ + \mathcal{E}^- = \mathcal{D}$. This decomposition of $\mathcal{D}$ defines the equations of motion (2.3). On the other hand, the decomposition $\mathcal{D} = \mathcal{G} + \tilde{\mathcal{G}}$ enables us to write down the equations of motion as those for a sigma model on G (resp. $\tilde{G}$) provided $\mathcal{E}^\pm$ are transversal to $\mathcal{G}, \tilde{\mathcal{G}}$, i.e. exists a linear invertible map

$$\mathbf{E} : \mathcal{G} \to \tilde{\mathcal{G}} \tag{3.1}$$

such that $\mathcal{E}^+ = \mathrm{span}\{t + \mathbf{E}(t), t \in \mathcal{G}\}$. Following [1, 2], we may decompose $l \in D$ in the vicinity of the group unit as

$$l = g.\tilde{h}\,, \qquad g \in G\,, \qquad \tilde{h} \in \tilde{G}$$

(which is always possible on Drinfeld doubles) and by eliminating $\tilde{h}$ from (2.3) one finds that the explicit form of the sigma model equations is given by the lagrangian

$$L = K_{ij}(g)(g^{-1}\partial_- g)^i (g^{-1}\partial_+ g)^j\,, \tag{3.2}$$

where

$$K(g) = (a(g) + E_0\, b(g))^{-1} E_0\, d(g)\,, \tag{3.3}$$

E_0 is a constant matrix and $a(g), b(g), d(g)$ are $n \times n$ submatrices of the adjoint representation of the group G on $\mathcal{D}$ in the basis $(X_i, \tilde{X}^j)$[1]

$$Ad(g)^t = \begin{pmatrix} a(g) & 0 \\ b(g) & d(g) \end{pmatrix}, \tag{3.4}$$

$$a(g)^{-1} = d(g)^t\,, \qquad b(g)^t a(g) = -a(g)^t b(g)\,. \tag{3.5}$$

An alternative formulation of the lagrangian in terms of right-invariant fields that we shall use in the following is

$$L = E_{ij}(g)(\partial_- g g^{-1})^i (\partial_+ g g^{-1})^j\,, \tag{3.6}$$

where

$$E(g) = (E_0^{-1} + \Pi(g))^{-1}\,, \qquad \Pi(g) = b(g) a(g)^{-1} = -\Pi(g)^t\,. \tag{3.7}$$

These forms of the lagrangian guarantee that the sigma model is classically dualizable because if it is rewritten in terms of derivatives of coordinates instead of invariant fields

$$L = F_{ij}(\phi)\partial_- \phi^i \partial_+ \phi^j\,, \tag{3.8}$$

[1] t denotes transposition.

where $\phi : \mathbf{R}^2 \longrightarrow \mathbf{R}^n$ then the covariant tensor field F on the manifold G

$$F_{ij} = e_i^a(g) E_{ab}(g) e_j^b(g)$$

satisfies the condition for dualization of the model

$$(\mathcal{L}_{v_c} F)_{ij} = \tilde{f}_c^{ab} v_a^m v_b^n F_{im} F_{nj} \tag{3.9}$$

where $v_a(g)$ are left-invariant vector fields that generate the right action G on itself, $e^b(g)$ are the dual left-invariant 1-forms,

$$e_i^a(g) v_a^j(g) = \delta_i^j \,, \qquad v_a^k(g) e_k^b(g) = \delta_a^b \,,$$

and $\tilde{f}$ are structure constants of a (dual) Lie group. The covariant tensor F can be understood as a sum of metric and torsion potential defining the geometric properties of the manifold G. Because (3.9) is a generalization of condition for a vector field being an isometry, the vector fields $v_a(g)$ are called generalized isometries.

The matrix E_0 corresponds to the choice of the decomposition $\mathcal{D} = \mathcal{E}^+ + \mathcal{E}^-$ and its form can be partially changed by the left translation of G. Namely, if $g \mapsto \hat{g} = hg$ then

$$\Pi(\hat{g}) = b(h) a^{-1}(h) + d(h) \Pi(g) a^{-1}(h)$$

and

$$F^{-1}(\hat{g}) = v^t(g) [\hat{E}_0^{-1}(h) + \Pi(g)] v(g)$$

where

$$\hat{E}_0^{-1}(h) = a^t(h) [E_0^{-1} a(h) + b(h)] \,. \tag{3.10}$$

One should also notice that although there are nonisomorphic Manin triples whose commutation relations differ just by overall multiplication constant b in all the commutators in the second subalgebra, e.g. the class $(3|3.i|b)$, such Manin triples lead to equivalent models. The reason is that such rescaling leads to

$$a(g) \to a(g) \,, \qquad b(g) \to b\, b(g) \,, \qquad d(g) \to d(g)$$

and consequently to rescaling of the lagrangian of such model

$$E_0 \to \frac{E_0}{b} \,, \qquad L \to \frac{L}{b} \,.$$

In the dual model rescaling by b^{-1} occurs. Therefore in the following we always put

$$b = 1$$

and don't mention b explicitly in designations of Manin triples.

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4. Poisson-Lie T-plurality

4.1 Classical level

As noted in [1, 2] and investigated explicitly in [9], the possibility to decompose some Drinfeld doubles into more than two Manin triples enables us to construct more than two equivalent sigma models and this property can be called Poisson-Lie T-plurality.

Let $\{T_j, \tilde{T}^k\},\ j,k \in \{1,\ldots,n\}$ be generators of the Lie algebras of a Manin triple associated with the lagrangian (3.6) and $\{U_j, \tilde{U}^k\}$ be generators of another Manin triple in the same Drinfeld double related by the $2n \times 2n$ transformation matrix:

$$\begin{pmatrix} T \\ \tilde{T} \end{pmatrix} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} U \\ \tilde{U} \end{pmatrix}. \tag{4.1}$$

The transformed model is then given by lagrangian of the same form as (3.6) but with $E(g)$ replaced by

$$E_u(g) = M(N + \Pi_u\, M)^{-1} \tag{4.2}$$

where

$$M = S^t E_0 - Q^t\,, \qquad N = P^t - R^t E_0\,, \tag{4.3}$$

and Π_u is calculated by (3.7) but from the adjoint representation of the group G_u generated by $\{U_j\}$ related to generators of G by (4.1). Using these formulas we can write down the explicit form of 22 classes of mutually classically equivalent three-dimensional sigma models.

Examples of Poisson-Lie T-dual models where the matrices in (4.1) are block diagonal resp. block antidiagonal

$$\begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$$

are sigma models (3.8) on the doubles $DD20$ and $DD21$ given by tensors[2]

$$F_{ij}(\phi) = (1 + (\phi^3)^2)^{-1} \begin{pmatrix} 1 & a\,\phi^3 & \phi^2 \\ -a\,\phi^3 & 1 & -a\,\phi^2\,\phi^3 \\ \phi^2 & a\,\phi^2\,\phi^3 & 1 + (\phi^3)^2 + (\phi^2)^2 \end{pmatrix} \tag{4.4}$$

$$\tilde{F}_{ij}(\phi) = (1 + (\phi^1)^2)^{-1} \begin{pmatrix} 1 + (\phi^1)^2 & 0 & 0 \\ 0 & 1 + (\phi^1)^2 & (1 + a)\,\phi^1 \\ 0 & (-1 + a)\,\phi^1 & 1 \end{pmatrix} \tag{4.5}$$

where $a = 1$ for the double $DD20$ and $a = -1$ for the double $DD21$. Both these models have nonzero and nonconstant torsion and scalar curvature.

The elements of G and $\tilde{G}$ are parametrized as

$$g = e^{\phi^1 X_1} e^{\phi^2 X_2} e^{\phi^3 X_3}\,, \qquad \tilde{g} = e^{\tilde{\phi}_1 \tilde{X}^1} e^{\tilde{\phi}_2 \tilde{X}^2} e^{\tilde{\phi}_3 \tilde{X}^3}\,.$$

This convention will be used further as well. In order not to overburden the reader with too complicated notation we shall always assume that ϕ's denote the coordinates on the group just under consideration. Therefore ϕ's in different models on the same Drinfeld double denote coordinates on different groups.

[2] We are afraid that the (II,II) model in [7] is not selfdual as it does not satisfy the equation (3.9).

4.2 Quantum level — conformal invariance

The equivalence on quantum level is more complicated [10, 11]. In quantum theory the duality or plurality transformation must be supplemented by a correction that comes from integrating out the fields on the dual group $\tilde{G}$ in the path integral formulation. In some cases it can be absorbed at the 1-loop level into the transformation of the dilaton field Φ. The dilaton field can be understood as an additional function on G that defines the non-linear sigma model and couples to scalar curvature of the manifold. The transformation of the tensor F then must be acompanied by the transformation of the dilaton [9]

$$\Phi_u = \Phi + \ln \operatorname{Det}(N + \Pi_u M) - \ln \operatorname{Det}(\mathbf{1} + \Pi E_0) + \ln \operatorname{Det} a_u - \ln \operatorname{Det} a \tag{4.6}$$

where Π_u, a_u are calculated as in (3.4), (3.7) but from the adjoint representation of the group G_u.

The condition that the theory is ultraviolate finite is equivalent to the conformal invariance of the model. That is expressed by the vanishing β-function equations.[3] At the one loop level these equations read

$$0 = R_{ij} - \bigtriangledown_i \bigtriangledown_j \Phi - \frac{1}{4} H_{imn} H_j^{mn} \tag{4.7}$$

$$0 = \bigtriangledown^k \Phi H_{kij} + \bigtriangledown^k H_{kij} \tag{4.8}$$

$$0 = R - 2 \bigtriangledown_k \bigtriangledown^k \Phi - \bigtriangledown_k \Phi \bigtriangledown^k \Phi - \frac{1}{12} H_{kmn} H^{kmn} \tag{4.9}$$

where covariant derivatives $\bigtriangledown_k$, Ricci tensor R_{ij} and scalar curvature R are calculated from the metric

$$G_{ij} = \frac{1}{2}(F_{ij} + F_{ji}) \tag{4.10}$$

which is also used for lowering and raising indices and

$$H_{ijk} = \partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij} \tag{4.11}$$

where

$$B_{ij} = \frac{1}{2}(F_{ij} - F_{ji})\,. \tag{4.12}$$

5. Conformally invariant sigma models and their duals

To choose from the huge variety of three dimensional sigma models we looked for those that satisfy the 1-loop vanishing β-function equations.

As opposed to the metric and torsion, the Poisson-Lie T-duality gives no hint what the possible dilaton fields are. That's why we have chosen the simplest possibility and tested which dualizable models satisfy the equations (4.7)–(4.9) for the dilaton field equal to

[3] To be more precise, we tacitly assume that there are other 23 dimensions, in which spectator fields live, as required by criticality of bosonic strings. Thus the term $D - 26$ that should appear in (4.9) vanishes but we suppose that these 23 extra dimensions form a space which is flat and completely decouples from the group G. It means that the spacetime is assumed to be $G \times \mathbf{R}^{23}$ (resp. $G \times \mathbf{R}^q \times T^{23-q}$) and coordinates on the flat component and corresponding components of tensors etc. don't appear in the equations (4.7)–(4.9).

zero. For technical reasons we have restricted ourselves to the models on solvable Drinfeld doubles and the diagonal input matrices

$$E_0 = \begin{pmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{pmatrix} \tag{5.1}$$

In addition to the already known abelian model we have found three models with zero dilaton namely those corresponding to the Manin triples $(6_0|1)$, $(7_0|1)$ and $(3|3.i)$. All these models are flat and torsionless but their duals do not share these properties.

5.1 Selfdual double DD 8

The first example of a model that satisfies the vanishing β-function equations (4.7)–(4.9) with $\Phi \equiv 0$ is that produced by the (only possible) decomposition of the double $DD8$ into $(3|3.i)$ and[4]

$$p = \pm 1\,, \qquad q = r = \pm k\,.$$

This gives[5]

$$F_{ij} = \pm k(e^{4\phi^1} + 2k\,\rho^2)^{-1} \begin{pmatrix} k^{-1}e^{4\phi^1} & -\rho & -\rho \\ \rho & \frac{1}{2}(1+e^{4\phi^1}) + k\,\rho^2 & \frac{1}{2}(1-e^{4\phi^1}) - k\,\rho^2 \\ \rho & \frac{1}{2}(1-e^{4\phi^1}) - k\,\rho^2 & \frac{1}{2}(1+e^{4\phi^1}) + k\,\rho^2 \end{pmatrix} \tag{5.2}$$

where

$$\pm\rho = \phi^2 + \phi^3\,.$$

In spite of the fact that the tensor field F is not symmetric the torsion H calculated by (4.11) and (4.12) vanishes as well as the Riemann tensor.

This model is classically selfdual, i.e. the tensor field $\tilde{F}_{ij}$ is obtained from F_{ij} only by $k \to 1/k$, but it is well known fact that models defined for Manin triples whose structure coefficients have nonzero trace $\tilde{f}_{ij}^{j}$ (which is the case of the Bianchi algebra **3**) are anomalous on the quantum level. It means that the effective action has a gravitational anomaly that cannot be absorbed into the transformation of the dilaton. If one nevertheless tries to construct the dual dilaton using (4.6), one finds that it depends on the coordinates on both subgroups. Since the dilaton must not depend on the subgroup integrated out in path integral, it is not well defined in this case. Such problem was encountered already in models constructed in [9] and its solution is still unknown, presumably it will require some modification of (4.6).

5.2 T-Duality in DD 15

The second example of a sigma model that satisfies the vanishing β-function equations with $\Phi \equiv 0$ is the model with the metric

$$F_{ij} = \begin{pmatrix} p & 0 & p\,\phi^2 \\ 0 & p & -p\,\phi^1 \\ p\phi^2 & -p\phi^1 & r + p\,(\phi^1)^2 + p\,(\phi^2)^2 \end{pmatrix} \tag{5.3}$$

[4] $\pm$ signs represent the same sign in all the following expressions.

[5] Recall that $F_{ij}(0) = E_{ij}(e) = (E_0)_{ij}$.

obtained from the decomposition of the Drinfeld double $DD15$ into $(7_0|1)$. The tensor field of the dual model corresponding to $(1|7_0)$ is

$$\tilde{F}_{ij} = \left[p\, r + (\phi^1)^2) + (\phi^2)^2\right]^{-1} \begin{pmatrix} r + \frac{1}{p}(\phi^1)^2 & \frac{1}{p}\phi^1\phi^2 & -\phi^2 \\ \frac{1}{p}\phi^1\phi^2 & r + \frac{1}{p}(\phi^2)^2 & \phi^1 \\ \phi^2 & -\phi^1 & p \end{pmatrix}. \tag{5.4}$$

The dual dilaton field calculated from (4.6) is

$$\tilde{\Phi}(\phi) = \ln\left(-p\, \left(r\, p + (\phi^1)^2 + (\phi^2)^2\right)\right) \tag{5.5}$$

and together with the tensor field (5.4) they satisfy the vanishing β-function equations so that the duality is preserved also on the quantum level.

Note that in the case of $p\, r < 0$ both the tensor and dilaton fields are singular for

$$(\phi^1)^2 + (\phi^2)^2 = -p\, r\,. \tag{5.6}$$

At these points curvatures are infinite, even the scalar one

$$R = 2p\frac{5p\, r - 2(\phi^1)^2 - 2(\phi^2)^2}{\left(p\, r + (\phi^1)^2 + (\phi^2)^2\right)^2}\,,$$

therefore there is a genuine singularity on the hypersurface (5.6) in the target manifold.

In the β-function equations the singularities of curvature, torsion and dilaton cancel provided one takes limits in (4.7)–(4.9).

By using transformation (4.1) transforming $(7_0|1)$ into isomorphic Manin triples, i.e. the same Manin triple $(7_0|1)$ but immersed in a different ways into $DD15$, one may also get models with nondiagonal matrix E_0 on the same group $G_{\mathbf{7_0}}$. By explicit calculation one finds the models with

$$E_0 = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ -e_{12} & e_{11} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix}, \tag{5.7}$$

where

$$e_{33} = \frac{4re_{11} + e_{32}^2 + 2e_{13}e_{31} + e_{31}^2 + e_{13}^2 + e_{23}^2 + 2e_{23}e_{32}}{4e_{11}}$$

and $e_{11}, e_{12}, e_{13}, e_{23}, e_{31}, e_{32}$ are arbitrary constants such that ${E_0}^{-1}$ exists, and nonzero but constant dilaton.[6] where $e_{11}, e_{12}, e_{13}, e_{23}, e_{31}, e_{32}, e_{33}$ are arbitrary constants such that ${E_0}^{-1}$ exists, and nonzero but constant dilaton. These models have nonvanishing torsion potential, but vanishing torsion and curvature. Therefore they differ from the original one only by closed, i.e. locally exact, form in lagrangian and constant dilaton shift and are consequently up to global issues equivalent to the original one (both on the classical and for most applications also on the quantum level). Similarly, one may also consider

[6]Note that consequently all models (5.4) with different choices of p are equivalent. Different choices of r give models which are not directly Poisson-Lie T-plural, but r can be absorbed into overall factor in the lagrangian using the fact that Manin triple $(7_0|1)$ is isomorphic to Manin triples $(7_0|1|b)$ where $\langle .,. \rangle$ is rescaled by b.

corresponding models on the dual group G_1 but again it seems that nothing qualitatively new emerges, only the expressions get rather complicated and not suitable for presentation here, e.g. the dilaton now up to the constant shift reads

$$\begin{aligned}&\ln(-e_{11}^2 e_{33} + e_{23}e_{32}e_{11} + e_{11}e_{23}\phi^1 - e_{11}e_{32}\phi^1 - e_{11}(\phi^1)^2 - e_{33}e_{12}^2 + \\ &\quad + e_{13}e_{12}e_{32} + e_{12}e_{13}\phi^1 + e_{12}e_{32}\phi^2 - e_{31}e_{12}e_{23} + e_{31}e_{12}\phi^1 + \\ &\quad + e_{31}e_{11}e_{13} + e_{31}e_{11}\phi^2 + e_{23}e_{12}\phi^2 - e_{11}e_{13}\phi^2 - e_{11}(\phi^2)^2)\,.\end{aligned}$$

5.3 Poisson-Lie T-plurality in DD 11

This is the most interesting example as this double can be decomposed into more than two dual Manin triples. It was investigated with a spectator dependent E_0 in [9]. Even if no spectator dependence is assumed then there is a sigma model with zero dilaton and diagonal E_0, namely that corresponding to the Manin triple $(6_0|1)$.

None of the models given by other decompositions of this Drinfeld double and diagonal E_0 satisfy the β-function equations with $\Phi \equiv 0$. Nevertheless, we can find sigma models with nonzero dilaton fields corresponding to the other decompositions using transformations between different Manin triples and check the β-function function equations. These models have nontrivial metrics, torsions and dilatons.

5.3.1 Decompositions $(6_0|1)$ and $(1|6_0)$

As mentioned, the simplest model comes from the semiabelian decomposition $(6_0|1)$ that produces sigma model with the metric

$$F_{ij} = \begin{pmatrix} p & 0 & p\,\phi^2 \\ 0 & -p & -p\,\phi^1 \\ p\phi^2 & -p\phi^1 & r - p\,(\phi^1)^2 + p\,(\phi^2)^2 \end{pmatrix} \tag{5.8}$$

that is flat (and torsionless) so that the model satisfies the β-function equations with the vanishig dilaton field. The tensor field of the dual model is

$$\tilde{F}_{ij} = \left[p\,r - (\phi^1)^2 + (\phi^2)^2\right]^{-1} \begin{pmatrix} r - \frac{1}{p}(\phi^1)^2 & \frac{1}{p}\phi^1\phi^2 & \phi^2 \\ \frac{1}{p}\phi^1\phi^2 & -r - \frac{1}{p}(\phi^2)^2 & -\phi^1 \\ -\phi^2 & \phi^1 & p \end{pmatrix}. \tag{5.9}$$

This model is neither flat nor torsionless, nevertheless the β-function equations (4.7)–(4.9) are satisfied for the dual dilaton field

$$\tilde{\Phi}(\phi) = \ln\left(p\left(r\,p - (\phi^1)^2 + (\phi^2)^2\right)\right) \tag{5.10}$$

calculated from (4.6). Note that, like in the $DD15$ case, both the tensor and dilaton fields are singular for

$$(\phi^1)^2 = p\,r + (\phi^2)^2\,. \tag{5.11}$$

At these points curvatures are infinite, even the scalar one

$$R = -2p\frac{5p\,r + 2(\phi^1)^2 - 2(\phi^2)^2}{(p\,r - (\phi^1)^2 + (\phi^2)^2)^2}\,,$$

therefore there is a genuine singularity on the hypersurface (5.11) in the target manifold. In the β-function equations the singularities of curvature, torsion and dilaton cancel.

By using transformation (4.1) transforming $(6_0|1)$ into isomorphic Manin triples, similarly as in $DD15$ case one may again get models with nondiagonal matrix E_0 on the same group $G_{\mathbf{6_0}}$. By explicit calculation one finds the models with

$$E_0 = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ -e_{12} & -e_{11} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix}, \tag{5.12}$$

where

$$e_{33} = \frac{4r e_{22} - e_{31}^2 + e_{32}^2 + 2e_{23}e_{32} - 2e_{31}e_{13} + e_{23}^2 - e_{13}^2}{4e_{22}}$$

and $e_{11}, e_{12}, e_{13}, e_{23}, e_{31}, e_{32}$ are arbitrary constants such that ${E_0}^{-1}$ exists, and constant dilaton.[7] These models again, as in $DD15$ case, have nonvanishing torsion potential, but vanishing torsion and curvature, and, as in in $DD15$ case, are up to global issues equivalent to the original one. When one considers the corresponding models on the dual group $G_{\mathbf{1}}$ it again seems that nothing really new and interesting emerges, only the expressions get rather complicated, e.g. the dilaton now reads (up to the constant shift)

$$\begin{aligned} &\ln(e_{32}e_{12}e_{13} - e_{23}e_{12}e_{31} - \phi^2 e_{32}e_{12} - \phi^2 e_{23}e_{12} + \phi^1 e_{13}e_{12} + \\ &\quad + e_{31}e_{12}\phi^1 - e_{11}(\phi^1)^2 - e_{12}^2 e_{33} + e_{33}e_{11}^2 + e_{11}(\phi^2)^2 + e_{11}e_{23}\phi^1 + \\ &\quad + e_{11}\phi^2 e_{31} - e_{31}e_{11}e_{13} - e_{11}\phi^2 e_{13} - e_{11}\phi^1 e_{32} + e_{32}e_{23}e_{11}). \end{aligned}$$

5.3.2 Decomposition $(5|2.i)$

The general form of the transformation matrix (4.1) from $(6_0|1)$ to $(5|2.i)$ is rather complicated and contains eight free real parameters. Nevertheless, using (4.1) and(4.3) one can calculate the sigma model corresponding to the decomposition $(5|2.i)$ and equivalent to the previous two. The model is given by the tensor field

$$F_{ij}(\phi) = \begin{pmatrix} r + \gamma Y(a,b)Y(c,d) & Y(a,d) & \gamma Y(a(\beta-1), d(\beta+1)) \\ Y(c,b) & \gamma^{-1} & \beta - 1 \\ \gamma Y(c(1+\beta), b(\beta-1)) & \beta+1 & \gamma(\beta^2-1) \end{pmatrix} \tag{5.13}$$

where

$$Y(a,b) = a\, e^{\phi^1} + b\, e^{-\phi^1}$$

and $a, b, c, d, \beta, \gamma$ are real parameters coming from the transformation matrix and p, r. Dilaton field is (rather surprisingly) constant because this model is again flat and torsionless. This is another example of a model with constant dilaton field but nondiagonal matrix $\tilde{E}_0 = MN^{-1} = F(0)$. This is also the only case we know where the metric is flat but the necessary condition for dualizability (3.9) is satisfied non-trivially — neither left nor right hand side of it vanishes *per se*, i.e. we consider an algebra of genuine generalized isometries even for trivial, flat metric.

One can calculate the model corresponding to $(2.i|5)$ as well but as the Bianchi algebra **5** is not traceless it does not satisfy the β-function equation (4.9).

[7]Note that consequently all models (5.8) with different choices of p are equivalent. Different choices of r give models which are not directly Poisson-Lie T-plural, but using the fact that Manin triple $(6_0|1)$ is isomorphic to Manin triples $(6_0|1|b)$ where $\langle .,. \rangle$ is rescaled by b one may extract r as an overall factor in the lagrangian and in this sense all models (5.8) and all their Poisson-Lie T-plurals are equivalent.

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5.3.3 Decomposition $(5.ii|6_0)$

The general form of the transformation matrix (4.1) from $(6_0|1)$ to $(5.ii|6_0)$ contains seven free real parameters and the general form of the sigma model is so complicated that it is hardly of any use and nearly impossible to display. We shall present here its very special form corresponding to the symmetric matrix

$$\tilde{E}_0 = \begin{pmatrix} r & -r & 0 \\ -r & -3r & 0 \\ 0 & 0 & 1/r \end{pmatrix}.$$

The model is given by the tensor field

$$F_{ij} = (-\lambda + 2\,\lambda^2 + 2\,\varrho - 2\,\lambda\,\varrho)^{-1} \begin{pmatrix} r\,\alpha & r\,\beta & \gamma \\ r\,\beta & r\,\epsilon & \delta \\ -\gamma & -\delta & \lambda/r \end{pmatrix} \tag{5.14}$$

where

$$\begin{aligned}
\rho &= 2e^{\phi^1} \\
\lambda &= e^{(\phi^1+\phi^2)} \\
\alpha &= (\lambda - \varrho)\,\left(-2 + \varrho\,\lambda - \lambda^2\right) \\
\beta &= -\left(-\varrho + \varrho^2 + 2\,\lambda - \varrho\,\lambda - \varrho\,\lambda^2 + \lambda^3\right) \\
\gamma &= \lambda\,\left(1 - \varrho + \lambda\right) \\
\delta &= \varrho + \lambda - 2\,\varrho\,\lambda + \lambda^2 \\
\epsilon &= -\left(\varrho^2 + 2\,\lambda^2 - 2\,\varrho\,\lambda^2 + \lambda^4\right)\lambda^{-1}\,.
\end{aligned}$$

The dilaton for this model is of the form

$$\Phi = \ln\left(\frac{1}{2}\,(2\,e^{\phi_1+\phi_2} - 4\,e^{\phi_1} + 4\,e^{-\phi_2} - 1)\right) \tag{5.15}$$

and together with the tensor field (5.14) satisfy the vanishing β-function equations so that the duality on the quantum level is preserved.

Again the metric, its curvature and the dilaton are singular for ϕ_1, ϕ_2 satisfying

$$2\,e^{\phi_1+\phi_2} - 4\,e^{\phi_1} + 4\,e^{-\phi_2} = 1\,.$$

and in the β-function equations the singularities cancel against each other.

In principle one can also calculate the model corresponding to $(6_0|5.ii)$ but cannot expect that it satisfies the β-function equations as the Bianchi algebra **5** is not traceless.

5.3.4 Note on $(5|1)$

The remaining decompositions of the Drinfeld double $DD11$ are the Manin triples $(5|1)$ and $(1|5)$. We must discard $(1|5)$ for the same reason as $(2.i|5)$ and $(6_0|5.ii)$ but one naturally expects that the decomposition $(5|1)$ may produce another conformally invariant sigma model. Unfortunately this is not true. The reason is that $\Pi_u = 0$ (it holds for any $(X|1)$) so that $E_u(g) = MN^{-1}$ and the matrix N calculated by (4.3) is singular for any transformation matrix from $(6_0|1)$ to $(5|1)$ (in spite of the fact that they contain again several free parameters). More generally speaking in this case the transversality condition (3.1) on $\mathcal{E}^{\pm}$ is not satisfied for any values of parameters p, r and any choice of Manin triple isomorphic to $(5|1)$.

The cases that the transversality condition is not satisfied occur for the Manin triples $(5|2.i)$ and $(5ii|6_0)$ as well but only for special choices of parameters in the transformation matrices (4.1). It is interesting that even in these cases the metrics of the models are not everywhere singular.

6. Conclusions

6.1 Difficulties encountered on the classical level

As we have seen, in construction of dual models one may encounter several difficulties. Firstly, on the classical level one might naively assume that one may go to any Manin triple in the considered Drinfeld double and construct models on the corresponding groups. This is true only in the generic case, for special value of parameters in E_0 the transversality condition (3.1) might not be satisfied for some decompositions into Manin triple (as seen in the case of Drinfeld double $DD11$ and Manin triple $(5|1)$). Unfortunately, these cases might just be the ones satisfying some other requirements on the models (e.g. conformal invariance). Also if one considers the case with nontrivial spectators then the matrix E_0 depends on them and the duality may break down for some specific values of the spectator fields, e.g. on some hyperplane in the 4-dimensional spacetime. We don't know any way to circumvent such obstacle if it occurs.

It follows that also the notion of modular space of Drinfeld double defined in [1] and investigated e.g. in [13, 14] should be made more precise. We realize that whether it is possible to construct a model on certain Manin triple of Drinfeld double or not depends via the transversality condition (3.1) on the subspace $\mathcal{E}^+$ or equivalently on the matrix E_0. It means that the modular space as the set of all equivalent Poisson-Lie T-dual models depends not only on the Drinfeld double, but also on the choice of the subspace[8] $\mathcal{E}^+$; for different $\mathcal{E}^+$ the modular spaces may be nonisomorphic. The results obtained before in [13, 14], where modular space was identified with the set of all decomposition of Drinfeld double into Manin triples, are therefore valid only for generic $\mathcal{E}^+$, i.e. such that the transversality condition is satisfied for all possible Manin triples in the Drinfeld double. For certain special matrices E_0 transformation to some Manin triples might be ruled out and the modular space is then only a subset of the generic one.

Also we have obtained singular metrics as duals to ordinary flat metric. This might look rather surprising because the manifolds considered are in fact Lie groups but it gives an explicit example of local nature of Poisson-Lie T-duality (and also of its special version,

[8] More precisely it depends on the equivalence class of these subspaces $\mathcal{E}^+$ with respect to $\langle .,. \rangle$-preserving automorphisms of $\mathcal{D}$; the equivalent subspaces give rise to isomorphic modular spaces that can be identified by change of basis in $\mathcal{D}$.

the so-called nonabelian or semiabelian T-duality, where $\tilde{\mathcal{G}}$ is abelian). From construction the duality transformation is guaranteed to exists only in some neighborhood of group unit element and encountering singularity signals that one has applied the present formalism of T-duality beyond its domain of applicability. Nevertheless, the resulting models still seem to be interesting examples of singular conformally invariant string backgrounds and might become well-defined duals in some future reformulation of Poisson-Lie T-duality. They might be interpreted as spacetimes with brane-like objects.

6.2 Dilaton and quantum Poisson-Lie T-duality

In the quantum case the dilaton field is known to cause problems. Although its transformation was constructed in [9] so that the 1-loop conformal invariance, i.e. vanishing β-function equations, is satisfied in the dual model, the dilaton might not be a well defined object in the dual theory at all. Namely it might dependent on the coordinates on the dual group presumably integrated out, as seen already in [9]. This occurs here for $(3|3.i)$ (which is in any case problematic because of nonvanishing trace). Also from a mathematical point of view one tends to believe that if the Poisson-Lie T-plurality is to be well defined also on the 1-loop quantum level, there should exist a better description of the dilaton, namely as an object on the whole double not depending on the concrete choice of Manin triples. The dilatons in respective dual theories shall then follow from it.

Also one may observe that in each of the classes of Poisson-Lie T-plural models on $DD11$ and $DD15$ we have one flat model with zero torsion potential and zero dilaton. (The metrics of $(6_0|1)$ and $(7_0|1)$ become constant and diagonal for coordinates χ obtained from the group element parametrization $g = e^{\chi_3 X_3} e^{\chi_2 X_2} e^{\chi_1 X_1}$). Since for the considerations in string theory the group plays only an auxiliary role, the important data are the target manifold and its metric (together with torsion potential and dilaton), for practical purposes all these models appear to be equivalent and are also equivalent to the model on $(1|1)$ with Minkowski metric. All equivalence is of course only local, as was already stressed, global issues are not at the present level of understanding covered by Poisson-Lie T-duality at all.

6.3 Future prospects

After sheding some light on complications occuring in study of 1-loop conformally invariant Poisson-Lie T-dual models we would like to express our opinion that in spite of the current, still very limited, understanding of quantum properties of Poisson-Lie T-duality it might already give us some practical results, e.g. can be used to generate rather nontrivial examples of conformally invariant string backgrounds. We hope that explicit examples presented here can be used as nontrivial "guinea pigs" in future attempts to improve quantum version of Poisson-Lie T-duality and that the improved version of it might avoid some of the pitfalls encountered in the current investigation.

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A. Bianchi algebras

It is known that any 3-dimensional real Lie algebra can be brought to one of 11 forms by a change of basis. These forms represent non-isomorphic Lie algebras and are conventionally known as Bianchi algebras. They are denoted by $\mathbf{1},\ldots,\mathbf{5}$, $\mathbf{6_a}$,$\mathbf{6_0}$, $\mathbf{7_a}$,$\mathbf{7_0}$,$\mathbf{8}$,$\mathbf{9}$ (see e.g. [15], in literature often uppercase roman numbers are used instead of arabic ones).The corresponding Lie groups we denote e.g. by $G_{\mathbf{1}}$. The list of Bianchi algebras is given in decreasing order starting from simple algebras.

$\mathbf{9}:\ [X_1,X_2]=X_3,\ [X_2,X_3]=X_1,\ [X_3,X_1]=X_2,$ (i.e. $so(3)$)

$\mathbf{8}:\ [X_1,X_2]=-X_3,\ [X_2,X_3]=X_1,\ [X_3,X_1]=X_2,$ (i.e. $sl(2,\mathbf{R})$)

$\mathbf{7_a}:\ [X_1,X_2]=-aX_2+X_3,\ [X_2,X_3]=0,\ [X_3,X_1]=X_2+aX_3,\ a>0,$

$\mathbf{7_0}:\ [X_1,X_2]=0,\ [X_2,X_3]=X_1,\ [X_3,X_1]=X_2,$

$\mathbf{6_a}:\ [X_1,X_2]=-aX_2-X_3,\ [X_2,X_3]=0,\ [X_3,X_1]=X_2+aX_3,\ a>0,\, a\neq 1,$

$\mathbf{6_0}:\ [X_1,X_2]=0,\ [X_2,X_3]=X_1,\ [X_3,X_1]=-X_2,$

$\mathbf{5}:\ [X_1,X_2]=-X_2,\ [X_2,X_3]=0,\ [X_3,X_1]=X_3,$

$\mathbf{4}:\ [X_1,X_2]=-X_2+X_3,\ [X_2,X_3]=0,\ [X_3,X_1]=X_3,$

$\mathbf{3}:\ [X_1,X_2]=-X_2-X_3,\ [X_2,X_3]=0,\ [X_3,X_1]=X_2+X_3,$

$\mathbf{2}:\ [X_1,X_2]=0,\ [X_2,X_3]=X_1,\ [X_3,X_1]=0,$

$\mathbf{1}:\ [X_1,X_2]=0,\ [X_2,X_3]=0,\ [X_3,X_1]=0,$

B. List of Manin triples

We present a list of Manin triples based on [12]. The label of each Manin triple, e.g. $(\mathbf{8}|\mathbf{5.ii}|\mathbf{b})$, indicates the structure of the first subalgebra $\mathcal{G}$, e.g. Bianchi algebra $\mathbf{8}$, the structure of the second subalgebra $\tilde{\mathcal{G}}$, e.g. Bianchi algebra $\mathbf{5}$; roman numbers i, ii etc. (if present) distinguish between several possible pairings $\langle .,. \rangle$ of the subalgebras $\mathcal{G},\tilde{\mathcal{G}}$ and the parameter b indicates the Manin triples differing by the rescaling of $\langle .,. \rangle$ (if such Manin triples are not isomorphic).

The Lie structures of the subalgebras $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are written out in mutually dual bases (X_1,X_2,X_3) and $(\tilde{X}^1,\tilde{X}^2,\tilde{X}^3)$ where a transformation was used to bring $\mathcal{G}$ to the standard Bianchi form (therefore its structure is given in appendix A and not listed here). Because of (2.2) this information specifies the Manin triple completely.

The dual Manin triples $(\mathcal{D},\tilde{\mathcal{G}},\mathcal{G})$ are not written explicitly but can be easily obtained by $X_j \leftrightarrow \tilde{X}^j$.

1. Manin triples with the first subalgebra $\mathcal{G} = \mathbf{9}$:

   $(\mathbf{9}|\mathbf{1})$: $[\tilde{X}^1, \tilde{X}^2] = 0, [\tilde{X}^2, \tilde{X}^3] = 0$, $[\tilde{X}^3, \tilde{X}^1] = 0$.

   $(\mathbf{9}|\mathbf{5}|\mathbf{b})$: $[\tilde{X}^1, \tilde{X}^2] = -b\tilde{X}^2$, $[\tilde{X}^2, \tilde{X}^3] = 0$, $[\tilde{X}^3, \tilde{X}^1] = b\tilde{X}^3$, $b > 0$.

2. Manin triples with the first subalgebra $\mathcal{G} = \mathbf{8}$:

   $(\mathbf{8}|\mathbf{1})$: $[\tilde{X}^1, \tilde{X}^2] = 0$, $[\tilde{X}^2, \tilde{X}^3] = 0$, $[\tilde{X}^3, \tilde{X}^1] = 0$.

   $(\mathbf{8}|\mathbf{5.i}|\mathbf{b})$: $[\tilde{X}^1, \tilde{X}^2] = -b\tilde{X}^2$, $[\tilde{X}^2, \tilde{X}^3] = 0$, $[\tilde{X}^3, \tilde{X}^1] = b\tilde{X}^3$, $b > 0$.

   $(\mathbf{8}|\mathbf{5.ii}|\mathbf{b})$: $[\tilde{X}^1, \tilde{X}^2] = 0$, $[\tilde{X}^2, \tilde{X}^3] = b\tilde{X}^2$, $[\tilde{X}^3, \tilde{X}^1] = -b\tilde{X}^1$, $b > 0$.

   $(\mathbf{8}|\mathbf{5.iii})$: $[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^2$, $[\tilde{X}^2, \tilde{X}^3] = \tilde{X}^2$, $[\tilde{X}^3, \tilde{X}^1] = -(\tilde{X}^1 + \tilde{X}^3)$.

3. Manin triples with the first subalgebra $\mathcal{G} = \mathbf{7_a}$:

   $(\mathbf{7_a}|\mathbf{1})$: $[\tilde{X}^1, \tilde{X}^2] = 0$, $[\tilde{X}^2, \tilde{X}^3] = 0$, $[\tilde{X}^3, \tilde{X}^1] = 0$.

   $(\mathbf{7_a}|\mathbf{2.i})$: $[\tilde{X}^1, \tilde{X}^2] = 0$, $[\tilde{X}^2, \tilde{X}^3] = \tilde{X}^1$, $[\tilde{X}^3, \tilde{X}^1] = 0$.

   $(\mathbf{7_a}|\mathbf{2.ii})$: $[\tilde{X}^1, \tilde{X}^2] = 0$, $[\tilde{X}^2, \tilde{X}^3] = -\tilde{X}^1$, $[\tilde{X}^3, \tilde{X}^1] = 0$.

   $(\mathbf{7_a}|\mathbf{7_{1/a}}|\mathbf{b})$: $[\tilde{X}^1, \tilde{X}^2] = b(-\frac{1}{a}\tilde{X}^2 + \tilde{X}^3)$, $[\tilde{X}^2, \tilde{X}^3] = 0$, $[\tilde{X}^3, \tilde{X}^1] = b(\tilde{X}^2 + \frac{1}{a}\tilde{X}^3)$, $b \in \mathbf{R} - \{0\}$.

4. Manin triples with the first subalgebra $\mathcal{G} = \mathbf{7_0}$:

   $(\mathbf{7_0}|\mathbf{1})$: $[\tilde{X}^1, \tilde{X}^2] = 0$, $[\tilde{X}^2, \tilde{X}^3] = 0$, $[\tilde{X}^3, \tilde{X}^1] = 0$.

   $(\mathbf{7_0}|\mathbf{2.i})$: $[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^3$, $[\tilde{X}^2, \tilde{X}^3] = 0$, $[\tilde{X}^3, \tilde{X}^1] = 0$.

   $(\mathbf{7_0}|\mathbf{2.ii})$: $[\tilde{X}^1, \tilde{X}^2] = -\tilde{X}^3$, $[\tilde{X}^2, \tilde{X}^3] = 0$, $[\tilde{X}^3, \tilde{X}^1] = 0$.

   $(\mathbf{7_0}|\mathbf{4}|\mathbf{b})$: $[\tilde{X}^1, \tilde{X}^2] = b(-\tilde{X}^2 + \tilde{X}^3)$, $[\tilde{X}^2, \tilde{X}^3] = 0$, $[\tilde{X}^3, \tilde{X}^1] = b\tilde{X}^3$, $b \in \mathbf{R} - \{0\}$.

   $(\mathbf{7_0}|\mathbf{5.i})$: $[\tilde{X}^1, \tilde{X}^2] = -\tilde{X}^2$, $[\tilde{X}^2, \tilde{X}^3] = 0$, $[\tilde{X}^3, \tilde{X}^1] = \tilde{X}^3$.

   $(\mathbf{7_0}|\mathbf{5.ii}|\mathbf{b})$: $[\tilde{X}^1, \tilde{X}^2] = 0$, $[\tilde{X}^2, \tilde{X}^3] = b\tilde{X}^2$, $[\tilde{X}^3, \tilde{X}^1] = -b\tilde{X}^1$, $b > 0$.

5. Manin triples with the first subalgebra $\mathcal{G} = \mathbf{6_a}$:

   $(\mathbf{6_a}|\mathbf{1})$: $[\tilde{X}^1, \tilde{X}^2] = 0$, $[\tilde{X}^2, \tilde{X}^3] = 0$, $[\tilde{X}^3, \tilde{X}^1] = 0$.

   $(\mathbf{6_a}|\mathbf{2})$: $[\tilde{X}^1, \tilde{X}^2] = 0$, $[\tilde{X}^2, \tilde{X}^3] = \tilde{X}^1$, $[\tilde{X}^3, \tilde{X}^1] = 0$.

   $(\mathbf{6_a}|\mathbf{6_{1/a}.i}|\mathbf{b})$: $[\tilde{X}^1, \tilde{X}^2] = -b(\frac{1}{a}\tilde{X}^2 + \tilde{X}^3)$, $[\tilde{X}^2, \tilde{X}^3] = 0$, $[\tilde{X}^3, \tilde{X}^1] = b(\tilde{X}^2 + \frac{1}{a}\tilde{X}^3)$, $b \in \mathbf{R} - \{0\}$.

   $(\mathbf{6_a}|\mathbf{6_{1/a}.ii})$: $[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^1$, $[\tilde{X}^2, \tilde{X}^3] = \frac{a+1}{a-1}(\tilde{X}^2 + \tilde{X}^3)$, $[\tilde{X}^3, \tilde{X}^1] = \tilde{X}^1$.

   $(\mathbf{6_a}|\mathbf{6_{1/a}.iii})$: $[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^1$, $[\tilde{X}^2, \tilde{X}^3] = \frac{a-1}{a+1}(-\tilde{X}^2 + \tilde{X}^3)$, $[\tilde{X}^3, \tilde{X}^1] = -\tilde{X}^1$.

6. Manin triples with the first subalgebra $\mathcal{G} = \mathbf{6_0}$:

   $(\mathbf{6_0}|\mathbf{1})$: $[\tilde{X}^1, \tilde{X}^2] = 0$, $[\tilde{X}^2, \tilde{X}^3] = 0$, $[\tilde{X}^3, \tilde{X}^1] = 0$.

   $(\mathbf{6_0}|\mathbf{2})$: $[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^3$, $[\tilde{X}^2, \tilde{X}^3] = 0$, $[\tilde{X}^3, \tilde{X}^1] = 0$.

   $(\mathbf{6_0}|\mathbf{4.i}|\mathbf{b})$: $[\tilde{X}^1, \tilde{X}^2] = b(-\tilde{X}^2 + \tilde{X}^3)$, $[\tilde{X}^2, \tilde{X}^3] = 0$, $[\tilde{X}^3, \tilde{X}^1] = b\tilde{X}^3$, $b \in \mathbf{R} - \{0\}$.

$(\mathbf{6_0}|\mathbf{4.ii})\ :\ [\tilde X^1,\tilde X^2]=(-\tilde X^1+\tilde X^2+\tilde X^3),\ [\tilde X^2,\tilde X^3]=\tilde X^3,\ [\tilde X^3,\tilde X^1]=-\tilde X^3\ .$

$(\mathbf{6_0}|\mathbf{5.i})\ :\ [\tilde X^1,\tilde X^2]=-\tilde X^2,\ [\tilde X^2,\tilde X^3]=0,\ [\tilde X^3,\tilde X^1]=\tilde X^3.$

$(\mathbf{6_0}|\mathbf{5.ii})\ :\ [\tilde X^1,\tilde X^2]=-\tilde X^1+\tilde X^2,\ [\tilde X^2,\tilde X^3]=\tilde X^3,\ [\tilde X^3,\tilde X^1]=-\tilde X^3.$

$(\mathbf{6_0}|\mathbf{5.iii}|\mathbf{b})\ :\ [\tilde X^1,\tilde X^2]=0,\ [\tilde X^2,\tilde X^3]=-b\tilde X^2,\ [\tilde X^3,\tilde X^1]=b\tilde X^1,\ b>0.$

7. Manin triples with the first subalgebra $\mathcal{G}=\mathbf{5}$:

$(\mathbf{5}|\mathbf{1})\ :\ [\tilde X^1,\tilde X^2]=0,\ [\tilde X^2,\tilde X^3]=0,\ [\tilde X^3,\tilde X^1]=0.$

$(\mathbf{5}|\mathbf{2.i})\ :\ [\tilde X^1,\tilde X^2]=0,\ [\tilde X^2,\tilde X^3]=\tilde X^1,\ [\tilde X^3,\tilde X^1]=0.$

$(\mathbf{5}|\mathbf{2.ii})\ :\ [\tilde X^1,\tilde X^2]=\tilde X^3,\ [\tilde X^2,\tilde X^3]=0,\ [\tilde X^3,\tilde X^1]=0.$

and dual Manin triples ($\mathcal{G}\leftrightarrow\tilde{\mathcal{G}}$) to Manin triples given above for $\mathcal{G}=\mathbf{6_0},\ \mathbf{7_0},\ \mathbf{8},\ \mathbf{9}$.

8. Manin triples with the first subalgebra $\mathcal{G}=\mathbf{4}$:

$(\mathbf{4}|\mathbf{1})\ :\ [\tilde X^1,\tilde X^2]=0,\ [\tilde X^2,\tilde X^3]=0,\ [\tilde X^3,\tilde X^1]=0.$

$(\mathbf{4}|\mathbf{2.i})\ :\ [\tilde X^1,\tilde X^2]=0,\ [\tilde X^2,\tilde X^3]=\tilde X^1,\ [\tilde X^3,\tilde X^1]=0.$

$(\mathbf{4}|\mathbf{2.ii})\ :\ [\tilde X^1,\tilde X^2]=0,\ [\tilde X^2,\tilde X^3]=-\tilde X^1,\ [\tilde X^3,\tilde X^1]=0.$

$(\mathbf{4}|\mathbf{2.iii}|\mathbf{b})\ :\ [\tilde X^1,\tilde X^2]=0,\ [\tilde X^2,\tilde X^3]=0,\ [\tilde X^3,\tilde X^1]=b\tilde X^2,\ b\in\mathbf{R}-\{0\}.$

and dual Manin triples ($\mathcal{G}\leftrightarrow\tilde{\mathcal{G}}$) to Manin triples given above for $\mathcal{G}=\mathbf{6_0},\ \mathbf{7_0}$.

9. Manin triples with the first subalgebra $\mathcal{G}=\mathbf{3}$:

$(\mathbf{3}|\mathbf{1})\ :\ [\tilde X^1,\tilde X^2]=0,\ [\tilde X^2,\tilde X^3]=0,\ [\tilde X^3,\tilde X^1]=0.$

$(\mathbf{3}|\mathbf{2})\ :\ [\tilde X^1,\tilde X^2]=0,\ [\tilde X^2,\tilde X^3]=\tilde X^1,\ [\tilde X^3,\tilde X^1]=0.$

$(\mathbf{3}|\mathbf{3.i}|\mathbf{b})\ :\ [\tilde X^1,\tilde X^2]=-b(\tilde X^2+\tilde X^3),\ [\tilde X^2,\tilde X^3]=0,\ [\tilde X^3,\tilde X^1]=b(\tilde X^2+\tilde X^3),$
$b\in\mathbf{R}-\{0\}.$

$(\mathbf{3}|\mathbf{3.ii})\ :\ [\tilde X^1,\tilde X^2]=0,\ [\tilde X^2,\tilde X^3]=\tilde X^2+\tilde X^3,\ [\tilde X^3,\tilde X^1]=0.$

$(\mathbf{3}|\mathbf{3.iii})\ :\ [\tilde X^1,\tilde X^2]=\tilde X^1,\ [\tilde X^2,\tilde X^3]=0,\ [\tilde X^3,\tilde X^1]=-\tilde X^1.$

10. Manin triples with the first subalgebra $\mathcal{G}=\mathbf{2}$:

$(\mathbf{2}|\mathbf{1})\ :\ [\tilde X^1,\tilde X^2]=0,\ [\tilde X^2,\tilde X^3]=0,\ [\tilde X^3,\tilde X^1]=0.$

$(\mathbf{2}|\mathbf{2.i})\ :\ [\tilde X^1,\tilde X^2]=\tilde X^3,\ [\tilde X^2,\tilde X^3]=0,\ [\tilde X^3,\tilde X^1]=0.$

$(\mathbf{2}|\mathbf{2.ii})\ :\ [\tilde X^1,\tilde X^2]=-\tilde X^3,\ [\tilde X^2,\tilde X^3]=0,\ [\tilde X^3,\tilde X^1]=0.$

and dual Manin triples ($\mathcal{G}\leftrightarrow\tilde{\mathcal{G}}$) to Manin triples given above for $\mathcal{G}=\mathbf{3},\ \mathbf{4},\ \mathbf{6_0},\ \mathbf{6_a},\ \mathbf{7_0},\ \mathbf{7_a}$.

11. Manin triples with the first subalgebra $\mathcal{G}=\mathbf{1}$:

$(\mathbf{1}|\mathbf{1})\ :\ [\tilde X^1,\tilde X^2]=0,\ [\tilde X^2,\tilde X^3]=0,\ [\tilde X^3,\tilde X^1]=0.$

and dual Manin triples ($\mathcal{G}\leftrightarrow\tilde{\mathcal{G}}$) to Manin triples given above for $\mathcal{G}=\mathbf{2}$–$\mathbf{9}$.

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